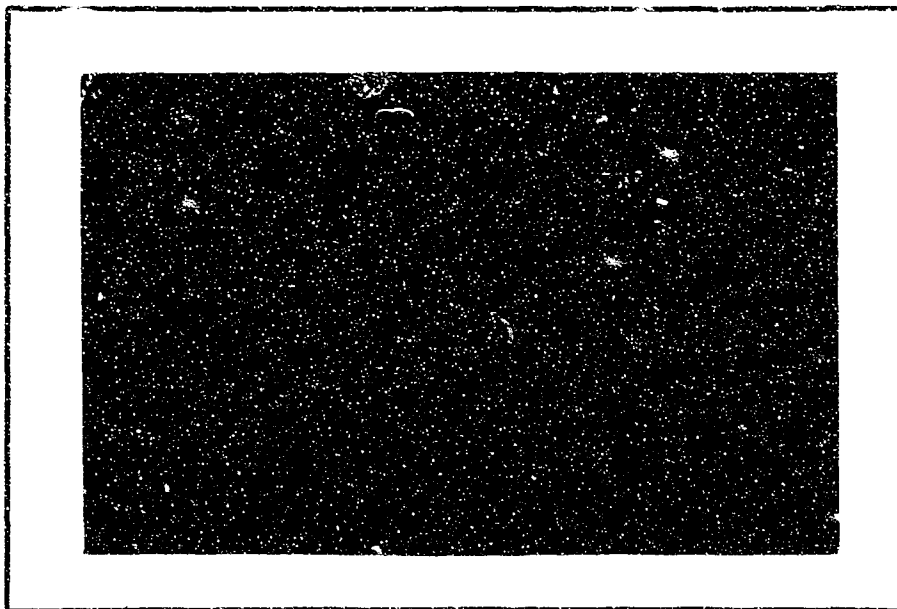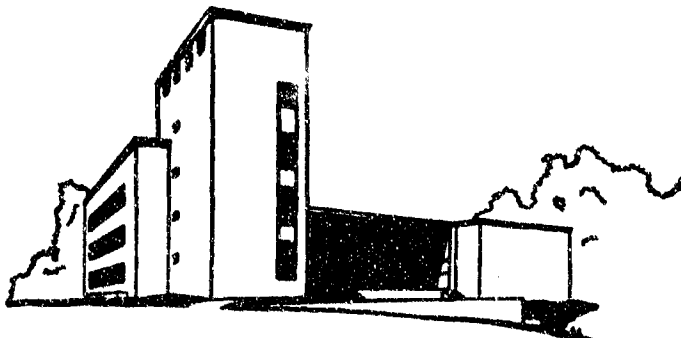# Carnegie-Mellon University

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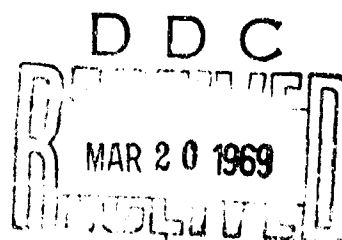## GRADUATE SCHOOL OF INDUSTRIAL ADMINISTRATION

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APPLICATIONS OF MATHEMATICAL CONTROL THEORY TO

ACCOUNTING AND BUDGETING

I.   THE CONTINUOUS WHEAT TRADING MODEL

by

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Abstract

A brief introduction to continuous mathematical control theory is presented first. Then a model having two state variable accounts, cash and wheat, is defined by means of differential equations. Adjoint functions, the Hamiltonian function, and the optimum (bang-bang) policy are derived. Interpretations of the Hamiltonian and switching functions and a numerical example are then given. Finally the results are extended to the case where transaction costs and spoilage costs are incurred.

APPLICATIONS OF MATHEMATICAL CONTROL THEORY TO

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I.   THE CONTINUOUS WHEAT TRADING MODEL

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1.   INTRODUCTION

In this paper we shall apply mathematical control theory to the
analysis of a simple model of an accounting and budgeting problem. Our ob-
jective is not only to make the model operational for practical applications
but also to use the framework and results of mathematical control theory in
order to get new insights into accounting and budgeting problems. In the
present paper we shall, however, limit our objective to the latter point,
leaving the former objective to our follow-up paper [3] which deals with the
discrete wheat trading model.

We shall be primarily concerned here with the process of the operations
of the firm looked at from an accounting standpoint. In accounting, a firm
is represented by a set of quantities that indicate the stocks of assets of
various types that are under the control of the firm, and the firm's
activities are indicated by the changes in these quantities. Using mathe-
matical notation we may say that the firm at time t has a vector $x(t)$
whose components indicate the physical amounts of each asset the firm has at
time t. The results of the activities of the firm between t_1 and t_2 are
measured by the vector difference $x(t_2) - x(t_1)$. From a conceptual point
of view it is convenient to consider derivative $\dot{x}(t)$ defined as,

$$\dot{x}(t) = \frac{dx(t)}{dt} = \lim_{t_2 \to t_1} \frac{x(t_2) - x(t_1)}{t_2 - t_1} \ .$$

Then we can say that the firm's activities at time t are indicated by $\dot{x}(t)$, which in turn affects $x(t)$.

The activities $\dot{x}(t)$ of a firm are aimed at changing the state $x(t)$ of the firm; but they are constrained by the state of the firm. For instance, the production rate at time t may be constrained by plant capacity at time t; interest paid out depends upon the amount of loans outstanding, etc. Such relationships between assets $x(t)$ and activities $\dot{x}(t)$ need not be fixed. They may be changed exogenously by factors outside the control of the firm, or they may be changed as the result of the intentional effort of the firm. For instance, plant deterioration occurs because of weather, machine wear and other natural causes, but it may be corrected by the firms engaging in repair activity. The prices of raw materials may be determined exogenously, but the firm's sale price may be partially or wholly at its own discretion.

When there is freedom in the asset-activity relationship that the management of the firm may exercise, it implies that management has responsibility to exercise the freedom in the most favorable way for the firm. It must answer questions such as: Should the plant be operated in two shifts? Should the plant be repaired? Or should the warehouse be modernized?

In answering these questions, it is necessary to take into account not only the production side of the problem, but simultaneously the marketing side, the finance side, the personnel side, etc. The accounting system offers a framework by which such aspects of the firm and its activities may be simultaneously considered. This point has already been demonstrated in the paper by Ijiri, Levy, and Lyon [2], in which linear programming techniques are

used in budgeting and financial planning based on the double-entry bookkeeping system.

The present paper is similar to the above paper [2] in approach and objective, however, the mathematical technique we shall employ here is that of mathematical control theory. In the next section we shall give a very brief introduction to this theory, but only enough to indicate the steps we are going to take. A reader who wishes to go into the reasons behind these steps should consult some of the references to mathematical control theory that appear in the bibliography. See [1, 4, 5].

2. AN OUTLINE OF THE MAXIMUM PRINCIPLE

We shall give a brief outline of the continuous maximum principle, in a form general enough for applications to the rest of the paper. Consider the problem

$$(1) \qquad \underset{x,m}{\text{Maximize}} \quad \int_0^T F(x,m,\tau)d\tau$$

subject to the differential equation constraints

$$(2) \qquad \dot{x}_i = f_i(x,m,t), \qquad x_i(0) = x_i^o$$

for $i = 1,\ldots,n$. Here the n-component (column) vector x consists of the state variables, and m is a k-component vector of control variables. It is assumed that m belongs to some prespecified set M of admissible controls, and we usually assume that M has certain properties, e.g., convexity. The functions F, f, and the initial values x^o are assumed known. The problem

is to choose a sequence of admissible control variables m from M in order that the resulting trajectory in the state space, determined by (2), maximizes the objective function (1).

We now proceed to put the problem in a certain standard form. Define a new variable,

$$(3) \qquad x_{n+1} = \int_0^t F(x,m,\tau)d\tau, \qquad x_{n+1}(0) = 0.$$

Then it is clear that

$$(4) \qquad \dot{x}_{n+1} = F(x,m,t) = f_{n+1}(x,m,t), \quad x_{n+1}(0) = 0 = x_{n+1}^o$$

This defines the function f_{n+1} and the value x_{n+1}^o. We now can restate the problem in (1) and (2) as

$$(5) \qquad \text{Maximize} \quad \delta x = x_{n+1}(T) , \quad \delta = (0,\ldots,0,1)$$

subject to

$$(6) \qquad \dot{x}_i = f_i(x,m,t) , \qquad x_i(0) = x_i^o$$

for $i = 1,\ldots,n,n+1$. This is the Mayer form of the problem. Note that the objective function is now a linear function of the state variables.

We now introduce n+1 adjoint variables (functions) z_i and require that they satisfy

$$(7) \qquad zx = \sum_{i=1}^{n+1} z_i x_i = k,$$

where k is a constant, for all values of t. Differentiating (7) with respect to time we obtain

$$(8) \qquad \dot{z}x + z\dot{x} = \sum_{i=1}^{n+1} (\dot{z}_i x_i + z_i \dot{x}_i) = 0$$

We now define the Hamiltonian function as

$$(9) \qquad \mathcal{H}(x,z,m,t) = -\dot{z}x = z\dot{x} = zf$$

where the second equality follows from (8) and the third equality follows from (6). Differentiating (9) with respect to x_i, yields

$$(10) \qquad \dot{z}_i = -\frac{\partial \mathcal{H}}{\partial x_i}$$

The boundary conditions for the adjoint variables will appear later. Note also that, differentiating (9) with respect to z_i we obtain

$$(11) \qquad \dot{x}_i = \frac{\partial \mathcal{H}}{\partial z_i}$$

Note that condition (7) is a kind of "perpendicularity" condition connecting the state variables and the adjoint variables. In many ways the adjoint variables play the same kind of role in control theory that the dual variables play in linear programming. However, adjoint variables, being functions over time, are inherently more complicated than dual variables.

We are now in a position to state the Pontryagin Maximum Principle, as follows: In order to solve the problem

$$(12) \qquad \text{Maximize } \delta x$$

subject to

$$(13) \qquad \dot{x}_i = \frac{\partial \mathcal{H}}{\partial z_i}, \qquad x(0) = x^0$$

a necessary condition is that there exist adjoint functions z satisfying,

(14)   Maximize $-zf = -\mathcal{H}$

subject to

(15) $\dot{z}_i = -\dfrac{\partial \mathcal{H}}{\partial x_i}$, $z(T) = -\delta$.

The reader is not expected to accept this result as obvious, and we are not going to produce a proof of the result here. However, we are going to use the above procedure to solve a management science problem which we call the continuous wheat trading model.

There are several important things to note about this theorem. First of all it gives necessary conditions only. Second it does not provide a computational method for finding the adjoint functions -- it merely gives a way of checking when they might be the correct ones. Third, notice that (13) is an initial value problem for the state variables, while (15) is a terminal value problem for the adjoint variables. Since we must have a simultaneous solution to both problems we have what is called in the literature a "two-point boundary value problem." Such problems have been found difficult to solve in general. Fortunately, in our case we shall be able to provide solutions.

5.   THE CONTINUOUS WHEAT TRADING MODEL

In order to obtain insight into how mathematical control theory can be used in accounting, let us consider the following example. A firm engages in buying and selling wheat. The firm's assets are of only these two types, cash and wheat, whose balances at time t are represented by $c(t)$ and $w(t)$, respectively, where $c(t)$ is measured in dollars and $w(t)$ in bushels of wheat. The price of wheat from time 0 to time T is assumed to be known in

advance with certainty and is given by a function $p(t)$. We shall let $P = p(T)$ be the terminal price of wheat. We assume that the initial assets $c(0) = C_0$ and $w(0) = W_0$ are known constants.

The firm's objective is to buy and sell wheat during the time 0 to T so as to maximize the total value of assets at time T, the end of the planning period. Hence it always values its current wheat inventory with the terminal price P, in making the trading decisions. Thus the total value of the firm's assets at time t, denoted by $x(t)$, is given by

(16) $x(t) = c(t) + Pw(t)$.

We denote the rate of buying or selling wheat by $m(t)$, expressed in bushels of wheat purchased if positive (or sold, if negative) per unit of time. We assume that $m(t)$ is constrained by

(17) $\underline{M} \leq m(t) \leq \overline{M}$ for $0 \leq t \leq T$,

where $\underline{M} < 0$ and $\overline{M} > 0$ are given constants. As long as this constraint is satisfied, the firm is permitted to be either long or short in either cash or wheat, or both. A long position is indicated by a positive stock of an asset and a short position by a negative stock.

There are three factors that change the cash balance. One is the interest received on cash (or paid on loan) balance at a given constant rate R per unit period, which is to be accrued continuously and to be received (or paid) in cash. That is, the interest on cash or loan balance is compounded continuously at rate R. The second factor is the inventory holding cost $h(w)$ that must be paid on an inventory of size $w(t)$ at time t. The

function h is defined for both positive and negative w and is assumed to be an increasing function of $|w|$. Thus, we interpret the inventory holding cost to be the cost of being short in wheat when $w(t) < 0$. The third factor is the cost (or revenue) $p(t)m(t)$ of purchases (or sales) of wheat at time t. Thus the differential equation that expresses the manner in which cash changes is

(18) $\dot{c}(t) = Rc(t) - h(w) - p(t)m(t)$.

Contrary to the cash balance, there is only one factor that changes the stock of wheat, namely the purchase or sale of wheat, $m(t)$ at time t. Thus the differential equation for w is

(19) $\dot{w}(t) = m(t)$.

To simplify the notation, we shall drop (t) from all expressions with the understanding that all lower case letters (except e) represent functions of time unless other independent variables are specified (e.g., h(w)), and that all upper case letters represent constants.

To put this problem into standard form we differentiate (16) obtaining

(20) $\dot{x} = \dot{c} + P\dot{w}$

$= Rc - h(w) - pm + Pm$

The latter expression was obtained by using (18) and (19)

Then the firm's objective is expressed by

(21)   Max $x(T) = c(T) + Pw(T)$

subject to differential equations (18), (19), and (20) and constraint (17).

The system of conditions (17)-(21) is called the <u>primal system</u>. The variables c, w, and x are <u>state variables</u>, and m is the <u>control variable</u>.

We want to choose a sequence of values for the control variable over time so as to maximize the terminal value of the firm's assets.

Following the outline of the previous section we introduce adjoint variables for each constraining equation. Specifically, we choose variable $\psi(t)$ for (18), $\emptyset(t)$ for (19), and the constant function -1 for (20). [If we had chosen an arbitrary function for (20) it is easy to show that it must be equal to -1 everywhere; we omit the proof of this.] Then the Hamiltonian function is

$$(22) \quad \mathcal{H} = \psi(Rc - h(w) - pm) + \emptyset m - (Rc - h(w) - pm + Pm)$$
$$= (\psi - 1)(Rc - h(w)) - [P + p(\psi-1) - \emptyset]m$$

We shall call the coefficient of m, namely,

$$(23) \quad s = P + p(\psi-1) - \emptyset$$

the switching function.

The maximum principle, previously discussed, provides as a necessary condition for an optimal policy that the control variable m should be chosen so as to maximize $-\mathcal{H}$. Since m occurs linearly in $-\mathcal{H}$, it follows that the optimal policy is bang-bang and is of the following form:

$$(24) \quad m = \begin{cases} \overline{M} & \text{if } s > 0 \\ \text{undetermined if } s = 0 \\ \underline{M} & \text{if } s < 0 \end{cases}$$

The case where s = 0 is called singular control. For most problems this occurs infrequently and can usually be ignored. We will give examples later.

Although the maximum principle gives only necessary conditions, the policy given in (24) is unique. Hence there is only one extremal policy which therefore must be the optimal policy. Hence, for the problem at hand the necessary conditions are also sufficient.

We now derive the differential equations for the adjoint functions. Differentiating $\mathcal{H}$ with respect to c we obtain

$$\dot{\psi} = -\frac{\partial \mathcal{H}}{\partial c} = -(\psi - 1)R$$

or

$$\dot{\psi} + R\psi = R$$

Multiplying through this equation by the integrating factor, e^{Rt}, integrating from t to T, and using the fact that $\psi(T) = 0$, we obtain

$$(25) \qquad \psi(t) = 1 - e^{R(T-t)}$$

As an interpretation of this function we note that the expression

$$1 - \psi = e^{R(T-t)}$$

is the future (time T) value of a dollar gained now (time t).

To obtain the differential equation for $\emptyset$ we differentiate $\mathcal{H}$ with respect to w;

$$(26) \qquad \dot{\emptyset} = -\frac{\partial \mathcal{H}}{\partial w} = (\psi - 1)h'(w) = -h'(w)e^{R(T-t)}$$

Integrating from t to T and using $\emptyset(T) = 0$ yields

$$(27) \qquad \emptyset(t) = \int_t^T h'(w)e^{R(T-t)}dt$$

The interpretation of $\emptyset(t)$ is that it is the total cost per unit of wheat of storing from t to T a small increment of wheat.

Let us now substitute these results back into the Hamiltonian function and give a new interpretation to $-\mathcal{H}$. We have

$$(28) \qquad -\mathcal{H} = [Rc - h(w)]e^{R(T-t)} + [P - pe^{R(T-t)} - \emptyset]m$$

We see that $\mathcal{H}$ has two terms, one not involving the control variable. The first term does not involve the control variable and represents the future

(time T) value of the interest Rc received on the cash account less the storage charges -h(w) on the wheat account at time t. The second term involves the switching function

$$(29) \qquad s = P - pe^{R(T-t)} - \emptyset$$

which can now be interpreted. It is the future (time T) price of wheat, P, less the future value $pe^{R(T-t)}$ of paying now (time t) for wheat at price p, and less the total storage cost $\emptyset$ from time t to T of the additional wheat bought now. The optimal policy given in (24) can now be stated as: buy (sell) the maximum amount at time t if it will yield a net profit (loss) at time T where the net profit or loss is to be evaluated taking into account all possible repercussions of the purchase or the sale decision. If this optimal policy is followed then the second term in (28) gives the total addition to future profits resulting from the optimal purchase (or sale) of wheat now.

We can now give a brief interpretation of $-\mathcal{H}$ that includes all of the foregoing. The function $-\mathcal{H}$ is the future (time T) value of the resent (time t) cash flow and consists of two parts: the first part is the cash flow due to previous (before time t) decisions, and the second part is the cash flow due to present (time t) decisions.

4. INVENTORY HOLDING COSTS

It may be interesting to see how the function $\emptyset(t)$, the future value of inventory holding cost, behaves under some specified forms of the function $h(w)$. First, let us consider the case where

$$(30) \qquad h(w) = Sw^2$$

Since the optimal policy given in (24) is bang-bang, namely either to buy at the maximum rate or to sell at the maximum rate,

$$(31) \qquad \dot{w} = m = M$$

for any t between two adjacent switching points, where M is a constant $\overline{M}$ or $\underline{M}$ depending upon the sign of the switching function. Remembering this, we substitute (30) into (27) and integrate (27) by parts, obtaining,

$$(32) \qquad \emptyset(t) = 2S \int_t^T we^{R(T-\tau)}d\tau$$

$$= 2S \left(-\frac{w}{R} - \frac{M}{R^2} \right)e^{R(T-\tau)} \Bigg|_t^T$$

$$= \frac{2S}{R} (we^{R(T-t)} - w(T)) + \frac{2SM}{R^2} (e^{R(T-t)} - 1)$$

for any t in the interval $t^* \quad t \leq T$, where t^* is the last switching point. By solving for $w(t)$ in a stepwise manner, we can then search for all switching points.

To show another example, let $h(w)$ be defined as follows:

$$(33) \qquad h(w) = S|w|$$

Then

$$(34) \qquad h'(w) = S \text{ sgn } w$$

where sgn w is 1 if $w > 0$, is 0 if $w = 0$, and is -1 if $w < 0$. Then,
for any t in $t** \leq t \leq T$ where $t**$ is the last time when the sign of
w is changed, the function $\emptyset(t)$ is derived as

$$(35) \qquad \emptyset(t) = h'(w) \int_t^T e^{R(T-\tau)} d\tau = h'(w)(-\frac{1}{R} e^{R(T-\tau)} \Big|_t^T$$

$$= \frac{h'(w)}{R} (e^{R(T-t)} - 1) = \frac{S \text{sgn } w}{R} (e^{R(T-t)} - 1)$$

Given the function $\emptyset(t)$, the switching function becomes:

$$(36) \qquad s = P - pe^{R(T-t)} - \frac{S \text{ sgn } w}{R} (e^{R(T-t)} - 1)$$

This may be written as

$$(37) \qquad s = e^{R(T-t)} [(\frac{S \text{ sgn } w}{R} + P)e^{-R(T-t)} - \frac{S \text{ sgn } w}{R} - p]$$

$$= e^{R(T-t)} (u-p)$$

where the function u is defined as

$$(38) \qquad u \equiv [\frac{S \text{ sgn } w}{R} + P]e^{-R(T-t)} - \frac{S \text{ sgn } w}{R}$$

Since the sign of s, which is of our interest, is equal to the sign of u - p
for $R \neq 0$ and finite T and t, we may draw a chart for u for given values of
R, S, T, P, and sgn w, and compare it with p to obtain sgn (u-p).

Let us consider the case where

R = .01 (1% per month compounded continuously)

S = .20 (dollars per unit per month)

T = 12 (month)

p(T)= 10 (dollars)

Then, if we let u^+, u^o, and u^- be the function u in which $\text{sgn } w = +1$, 0, and -1, respectively, they may be written as follows:

$$u^+ = 30e^{-R(T-t)} - 20$$

$$u^o = 10e^{-R(T-t)}$$

$$u^- = 10e^{-R(T-t)} + 20$$

In Figure 1, three curves are prepared for these functions.[1] It can be easily seen from (38) that for a larger inventory holding cost S, the spreads between u^+ and u^o as well as u^o and u^- become wider for any t except $t = T$, and that for a larger interest rate R, the same spreads become smaller and all three curves become more convex starting from lower points at $t=0$.

The curves in Figure 1 show the switching criterion in the following manner. Let us suppose that the sign of inventory balance does not change throughout the entire period of $o \leq t \leq T$. Then u^+ or u^- in Figure 1 is used to compare with p. If p is below u (u^+ or u^- depending upon sgn w) for any time interval $t_1 \leq t \leq t_2$, this means that sgn (u-p) (hence sgn s) is positive. Thus, by (24), m is set equal to $\overline{M}$ during this period, where $\overline{M}$ is the maximum purchase rate. On the other hand, if p lies above u for any time interval, m is set equal to $\underline{M}$ during this period, where $\underline{M}$ is the maximum sales rate. If p coincides with u for any time interval, the total

[1] For $0 \leq x \leq .12$, e^{-x} can be approximated by

$$e^{-x} \doteq 1 - .95x$$

with errors less than .0013 in the absolute values. The curves in Figure 1 are drawn based on this approximation.
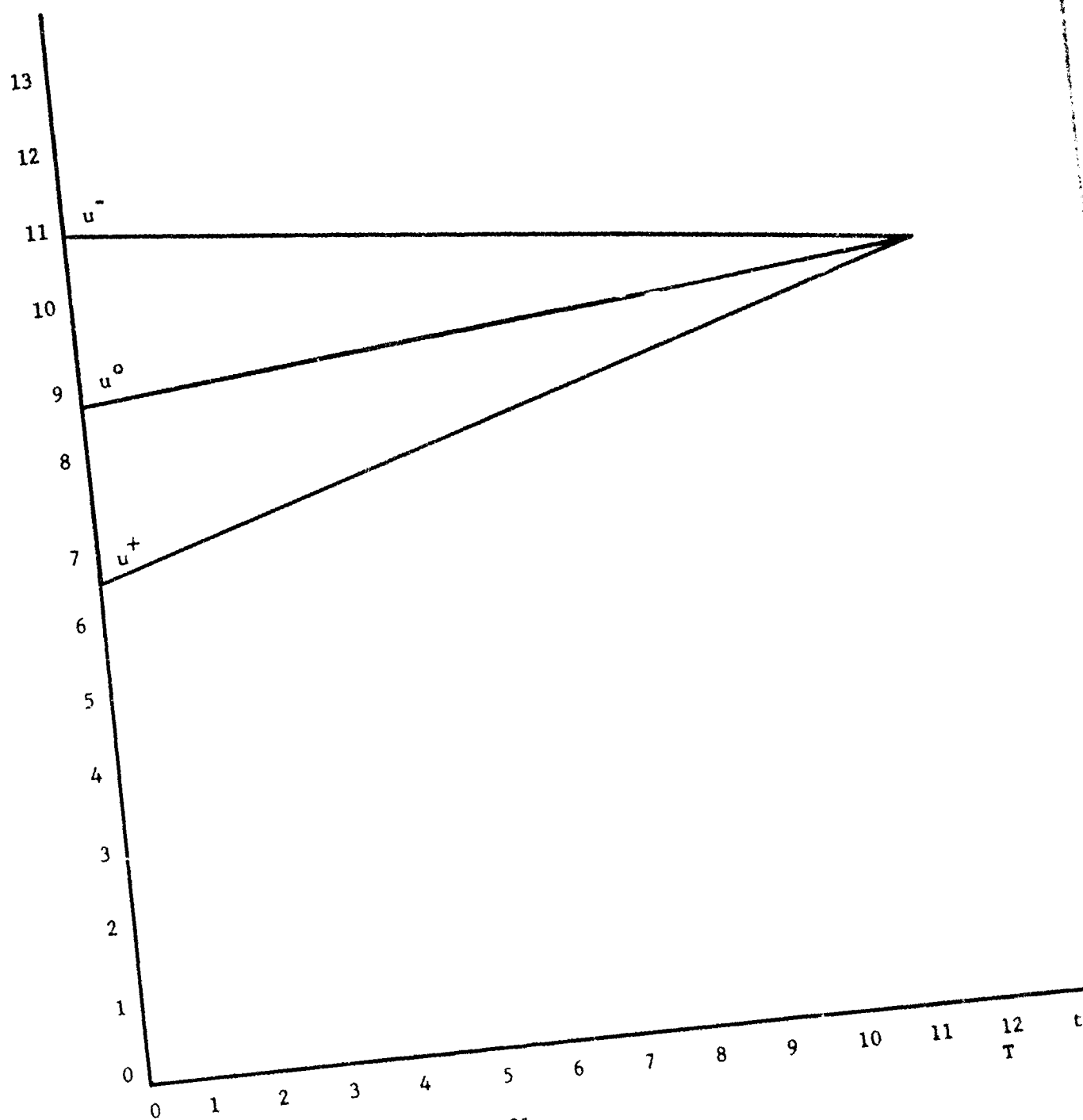
R = .01
S = .20
T = 12
P = 10

FIGURE 1

value of the firm's assets at time T is unaffected whether the firm buys or sells wheat or do nothing during this period, although the amount of the inventory balance at time T is certainly affected.

Figure 2 shows some examples of p together with indicated decisions on m.

Now let us consider situations where the sign of w changes during the period of $0 \leq t \leq T$. Consider the case (c) in Figure 2 where $p = 7.5$ for $0 \leq t \leq 6$. Suppose that the maximum purchase and sales rates ($\bar{M}$ and $\underline{M}$) are set equal to 1 and -1, respectively, and that the firm has only 2 units of wheat at $t = 0$. Then at $t = 2$, the inventory balance becomes 0, hence we must compare p with u^o rather than u^+. Since p lies below u^o, we must buy (rather than sell) wheat. However, as soon as we start buying wheat, w becomes positive; hence p must be compared with u^+ which lies below p, indicating "sell". On the other hand, we cannot keep selling, since as soon as w becomes negative, p must be compared with u^- which indicates "buy". Thus, once it is reached, $w = 0$ is an equilibrium point for any price between $u^+ \leq p \leq u^-$ at any time t. If the initial inventory is 0 and p lies entirely between u^+ and u^-, there will be no purchases or sales of wheat during the entire period. This situation is shown in Figure 3 by an example.

The analysis as elaborated in the above, may be difficult to obtain if the inventory holding cost is expressed as a more complicated function of w, since the switching function s may become a function of w, rather than just the sign of w. However, such a problem can be solved at least numerically by means of using an analogue computer or by means of transforming the problem into a discrete problem as elaborated in our paper [3] and the solution obtained by means of a digital computer.
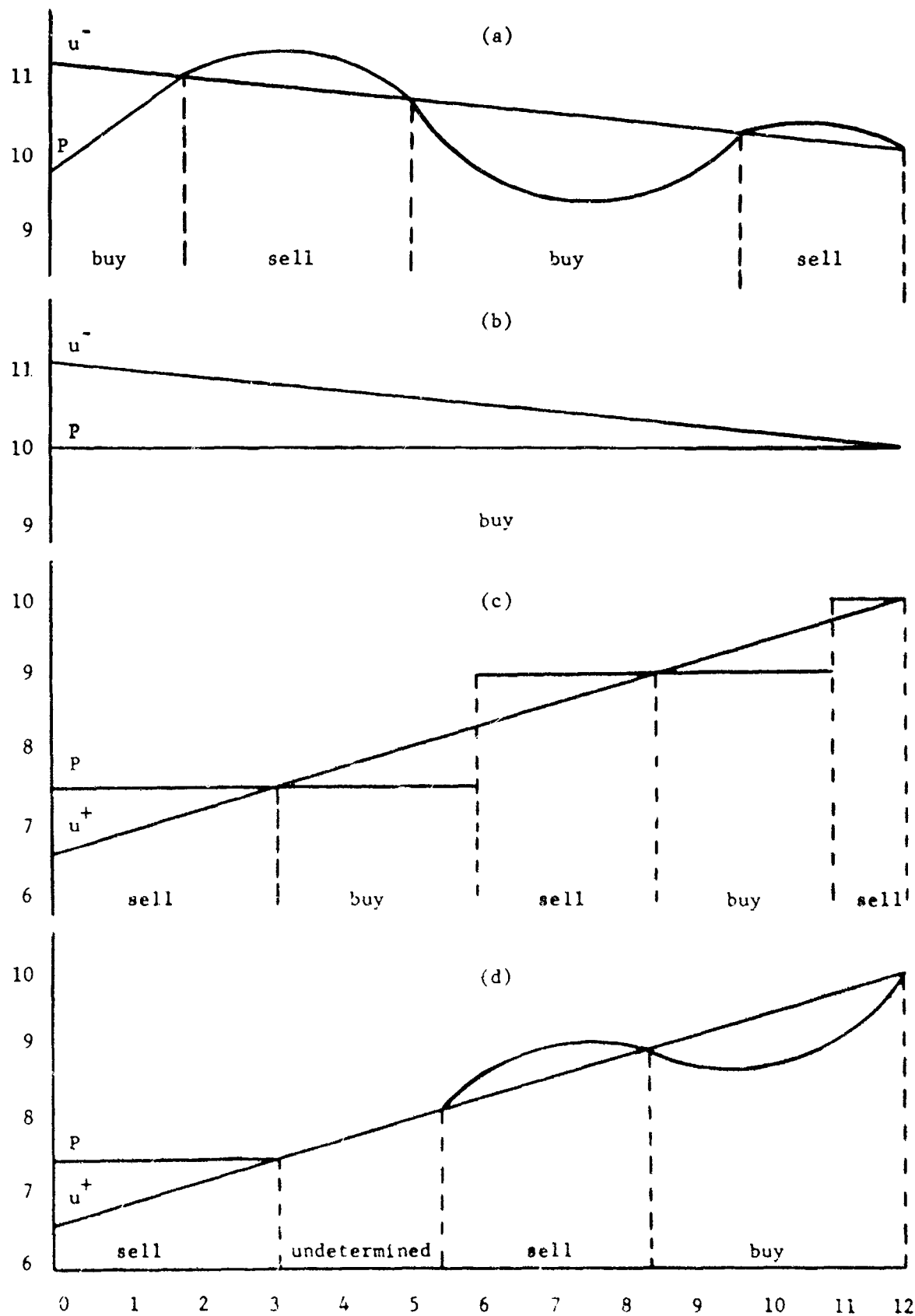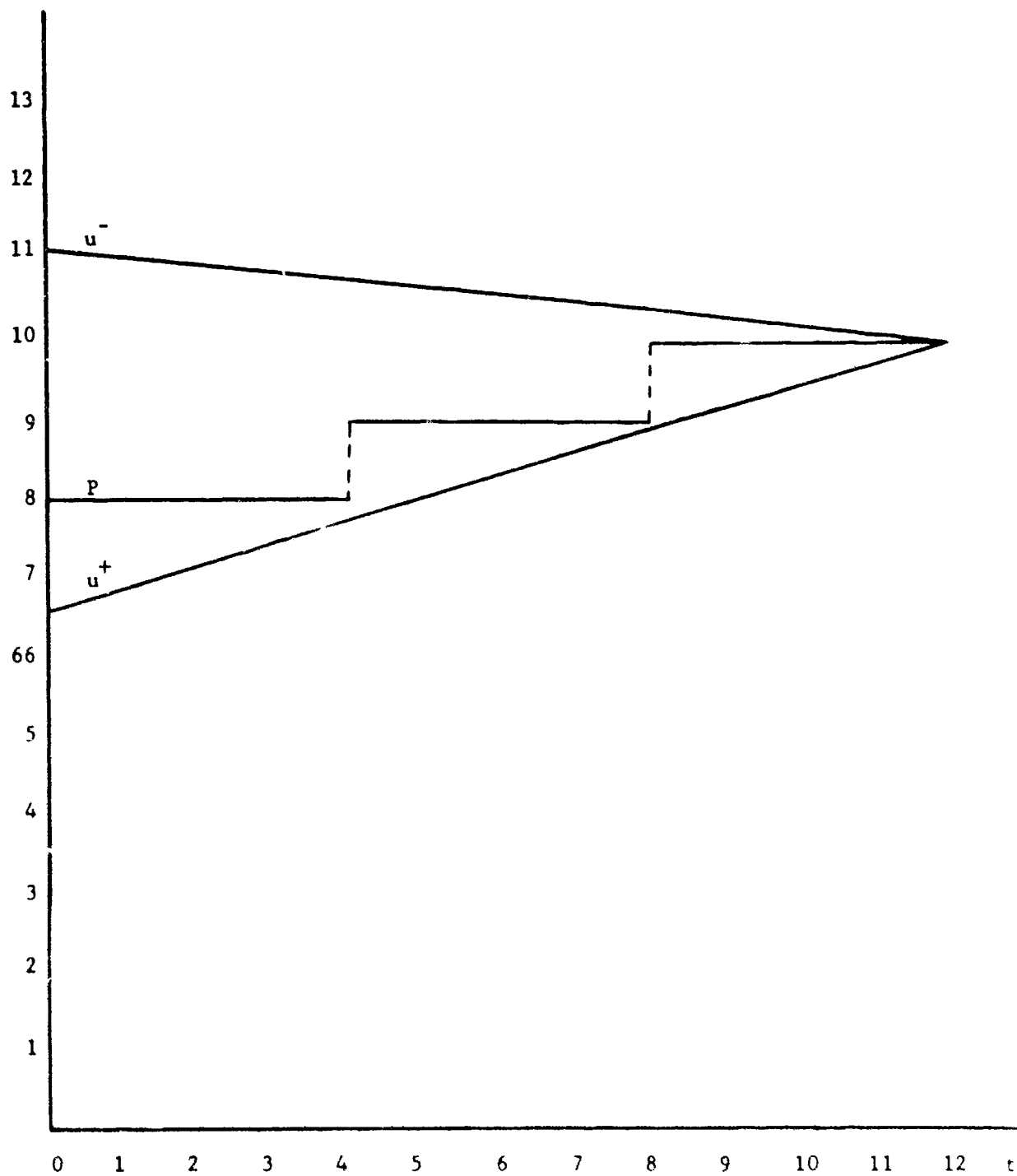
Figure 2

Figure 3

5. TRANSACTION AND SPOILAGE COSTS

In this section we work out a couple of minor variations in the model. First we assume that there is a constant proportion of the wheat that spoils. The rate of spoilage is given by a positive constant A. Second we assume that a transaction of size m incurs a unit transaction cost f(m), where f is a monotone increasing function for all values of m with f(0) = 0. Later on we shall assume that f is unbounded above and below, and show that we may drop the bounds (17) when this is true. Further, we shall assume throughout this section, w ≥ 0 for all t in the range considered.

We start by retaining equations (16) and (17). The rate of change of cash is

$$(39) \qquad \dot{c} = Rc - h(w) - [p + f(m)]m$$

the change from (18) being only in the addition of the transaction cost f(m)m. Note that the transaction cost f(m)m ≥ 0. Similarly, the rate of change of wheat is given by

$$(40) \qquad \dot{w} = m - Aw.$$

the only change from (19) is in the spoilage term, -Aw.
Substituting these into x, we have

$$(41) \qquad \dot{x} = \dot{c} + P\dot{w} = Rc - h(w) - PAw - [p + f(m) - P]m$$

From these and the same adjoint functions as before we have

$$(42) \qquad -\mathcal{H} = (1 - \psi)[Rc - h(w)] + (\emptyset - P)Aw + [P - \emptyset - (1-\psi)(p+f(m))]m$$

The differential equation for ψ is exactly the same as before, and hence ψ is given by (25). From this we obtain an expression for $-\mathcal{H}$ that is,

$$(43) \qquad -\mathcal{H} = [Rc - h(w)]e^{R(T-t)} + (\emptyset - P)Aw + [P-\emptyset-e^{R(T-t)}(p+f(m))]m$$

which has similar interpretations as before. Namely, the first term in (43) is the current change in cash due to past decisions; the second term is the current loss due to spoilage; the last term involves the switching function

$$(44) \qquad s = P - \emptyset - e^{R(T-t)}(p + f(m))$$

which is just like (29) with transaction costs added in. Thus the decision as to whether or not to buy now takes into account the transaction cost. The optimal policy is again bang-bang and is described in (24), provided $\underline{N}$ and $\overline{M}$ are not too large in absolute value. We return to this point later.

The derivation of the adjoint function $\emptyset$ is a little more complicated. It's differential equation is

$$(45) \qquad \dot{\emptyset} - A\emptyset = - h'(w)e^{R(T-t)} - PA$$

Multiplying by the integrating factor e^{-At}, integrating from t to T, and using the boundary condition $\emptyset(T) = 0$ yields

$$(46) \qquad \emptyset(t) = e^{At+RT} \int_{t}^{T} h'(w)e^{-(A+R)\tau}d\tau + P(1 - e^{-A(T-t)})$$

The first term is just like (27), except that the interest rate R has been augmented by the spoilage rate A. (In this model, we are, in effect, being charged for storing the spoiled wheat; that is an indirect storage cost that appears here as an increase in interest rate.) The second term of (46) measures the actual net weight of wheat taking into account the spoilage that will occur between time t and T if we have a bushel of wheat now.

Let us now return to the switching function (44) and the form of the optimal policy. Assume that $f(m)$ is unbounded above and below and is monotone increasing (it need not be continuous), and also $f(0) = 0$. Suppose

that we do not have bounds (17) on m. In Figure 4 we plot the situation

for two different prices p* with $p*e^{R(T-t)} < P - \emptyset$ and p** with

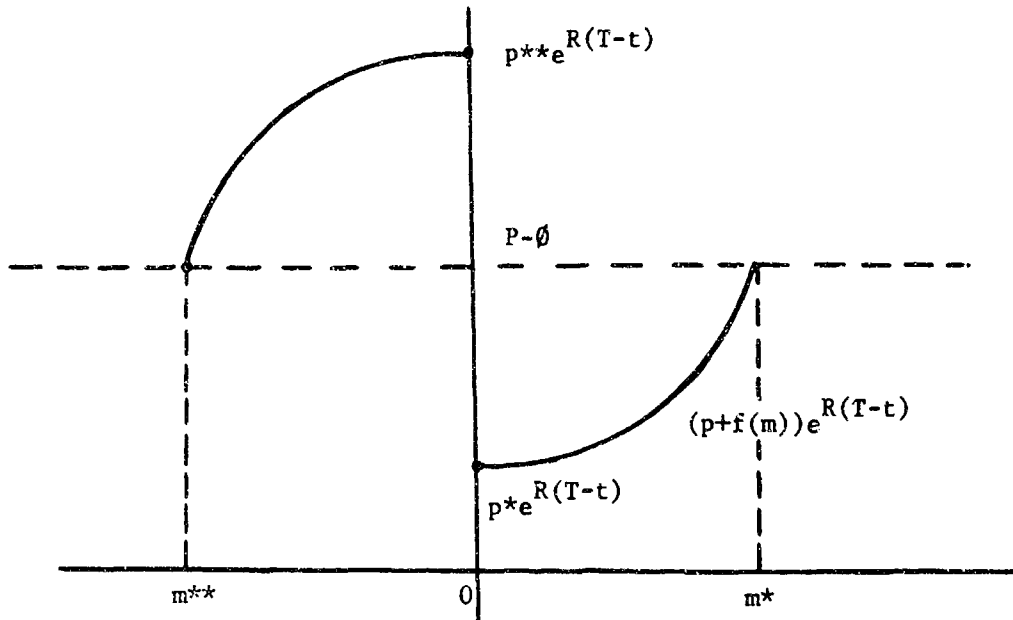$p**e^{R(T-t)} > P - \emptyset$. In the first case, it clearly pays to buy wheat and

Figure 4

the optimal strategy is buy m* bushels of wheat where m* solves the

equation

$$(47) \qquad (p + f(m))e^{R(T-t)} = P - \emptyset$$

In the second case, the optimal strategy is to sell m** units where m** is

the solution of (in this case the negative of) (47). The fact that there are

solutions of (47) in each case follow from the assumption that f(m) is un-

bounded above and below.

In the case in which (17) holds as well it is necessary to compare the

value m* with $\overline{M}$ and choose whichever is smaller to determine the optimal

purchase; and also compare m** with M and choose whichever is larger
(least negative) to determine the optimal sale.

6. CONCLUSIONS

We have demonstrated in the above how mathematical control theory may be
profitably applied to the overall operations of a firm via the accounting and
budgeting system, although the example we used is an extremely simple one in
order to obtain the theoretical insight in the clearest manner. It is cer-
tainly true that as we increase the number of variables (or accounts in the
accounting sense) the problem becomes harder and harder to solve. It is also
clear that for practi 1 applications, we must formulate the problem in a
discrete manner in order that it may be solved on a computer. However, we
are leaving these problems to our follow-up paper [3] as we mentioned at the
beginning. In the present paper, we are primarily interested in the
theoretical clarification of the problem.

As a result of applying the mathematical control theory, not only can
we develop a schedule for optimal control in terms of the values of control
variables but also we can prepare optimal projected financial statements
(or budgets) similar to the one discussed in the paper by Ijiri, Levy, and
Lyon [2]. Then any departure of the actual from the planned activities may
be analyzed in terms of errors in the model and suboptimal performances.

There is one additional factor derived from the model that is of
significant interest from the accounting standpoint. That is the accounting
meaning of the adjoint variables. There have been several attempts in
accounting valuation theory to use the discounted future cash-flow as a basis

of asset valuation. As we have seen above, the adjoint variables offer much richer basis of valuation, since it takes into account not only interest but also the effect of an incremental unit of an asset upon the objective of the firm taking into account the entire interaction of the firm's activities. We are, of course, not advocating the use of the adjoint variables as a substitute for all valuation basis; however, they do provide useful information for the planning purposes of the management, in an analogous way as the dual evaluation in linear programming does.

For instance, in the case discussed in Section 2, the adjoint variable ψ that is attached to the variable c for cash provides a criterion whether a further investment in this firm is more profitable than, say, an investment in a bond which will yield at a certain rate. A similar comparison may be made by the use of the adjoint variable for wheat. Furthermore, the switching function s may be used as a criterion for expanding the upper and lower bound for wheat transactions since the terminal asset value is increased by $|s|$ if the bound is extended by one unit.

We are far from having exploited this fruitful area of applications. This paper is only the beginning of many interesting and useful concepts and techniques that may be derived from the applications of the mathematical control theory in accounting and budgeting.

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13. ABSTRACT

A brief introduction to continuous mathematical control theory is presented first. Then a model having two state variable accounts, cash and wheat, is defined by means of differential equations. Adjoint functions, the Hamiltonian function, and the optimum (bang-bang) policy are derived. Interpretations of the Hamiltonian and switching functions and a numerical example are then given. Finally the results are extended to the case where transaction costs and spoilage costs are incurred.

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